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Solitary wave solution for the generalized Kawahara equation

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ABSTRACT

The travelling wave ansatz is used to find the solitary wave solution of the generalized Kawahara equation. The ansatz is obtained from the structure of the soliton solution of the Kawahara equation and the modified Kawahara equation. The first two integrals of motion of the generalized Kawahara equation are also computed in this work.

1. Introduction

The study of the theory of nonlinear evolution equations has come a long way, with the integrability aspects [1–10]. Starting from the classical technique of the Inverse Scattering Transform (IST), which in principle is the nonlinear analog of the Fourier transform, there are a bunch of new techniques for carrying out the integration of various nonlinear evolution equations. These new methods include the Wadati trace method, pseudo-spectral method, tanh–sech method, sine–cosine method and Riccati equation expansion method, and many more. Without these modern methods of integrability, many such equations would not have been solved. Although, once upon a time, the method of IST had a monopoly, these nonlinear evolution equations cannot be integrated using the IST as the corresponding Painlevé test will fail. It is to be noted that one of the major disadvantages of these modern methods of integrability is that one can only obtain the one-soliton solution of such a nonlinear evolution equation and not a multi-soliton solution. Also, these methods are unable to compute a solution for the soliton radiation. In this work, the more common method of travelling wave ansätze will be exploited to find the one-soliton solution of the generalized Kawahara equation (gKE).

The Kawahara equation (KE) and the modified Kawahara equation (mKE) are respectively given by

$$q_t + aqq_x + bq_{xxx} - cq_{xxxxx} = 0 \tag{1}$$

and

$$q_t + aq^2q_x + bq_{xxx} - cq_{xxxxx} = 0 \tag{2}$$

whose solutions are given in terms of sech^4 and sech^2 functions respectively [7,8]. Here, in (1) and (2), a, b and c are positive parameters. Now, the gKE that is going to be studied in this work is given by

$$q_t + aq^pq_x + bq_{xxx} - cq_{xxxxx} = 0 \tag{3}$$

where $p > 0$. It is therefore natural to take the ansatz of the one-soliton solution of the gKE in terms of the $\text{sech}^{4/p}$ function. Thus, the hypothesis of the one-soliton solution of the gKE is going to be given by

$$q(x, t) = \frac{A}{\cosh^{\frac{4}{p}} B\left(x - \bar{x}(t)\right)} \tag{4}$$

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where A represents the amplitude of the soliton, while B is the inverse width of the soliton and $\bar{x}$ represents the center position of the soliton and therefore the velocity of the soliton is given by [1]

$$v = \frac{d\bar{x}}{dt}. \tag{5}$$

2. Mathematical analysis

In order to obtain the one-soliton solution of the gKE the solution ansatz, given by (4), is substituted into the gKE, given by (3). In order to substitute (4) into (3), each of the terms in (3) is computed to be as follows:

$$q_t = -\frac{4vAB}{p}\frac{\tanh\tau}{\cosh^{\frac{4}{p}}\tau} \tag{6}$$

$$aq^p q_x = -\frac{4aA^{p+1}B}{p}\frac{\tanh\tau}{\cosh^{\frac{4}{p}+4}\tau} \tag{7}$$

$$bq_{xxx} = -\frac{64bAB^3}{p^3}\frac{\tanh\tau}{\cosh^{\frac{4}{p}}\tau} + b\left(\frac{4}{p}+2\right)\left(\frac{4AB^3}{p} + \frac{16AB^3}{p^2}\right)\frac{\tanh\tau}{\cosh^{\frac{4}{p}+4}\tau} \tag{8}$$

$$\begin{aligned} cq_{xxxxx} = {} & -\frac{1024cAB^5}{p^5}\frac{\tanh\tau}{\cosh^{\frac{4}{p}}\tau} + c\left(\frac{4}{p}+2\right)\left\{\left(\frac{256AB^5}{p^4} + \frac{64AB^5}{p^3}\right) + \left(\frac{4}{p}+2\right)^2\left(\frac{4AB^5}{p} + \frac{16AB^5}{p^2}\right)\right\}\frac{\tanh\tau}{\cosh^{\frac{4}{p}+2}\tau} \\ & - c\left(\frac{4}{p}+4\right)\left(\frac{4}{p}+2\right)^2\left(\frac{4AB^5}{p} + \frac{16AB^5}{p^2}\right)\frac{\tanh\tau}{\cosh^{\frac{4}{p}+4}\tau} - c\left(\frac{4}{p}+6\right)\left(\frac{4}{p}+2\right)\left(\frac{4AB^5}{p} + \frac{16AB^5}{p^2}\right)\frac{\tanh\tau}{\cosh^{\frac{4}{p}+6}\tau} \end{aligned} \tag{9}$$

where $\tau = B\left(x - \bar{x}(t)\right)$. Now substituting (6)–(9) into (3) yields the relations that are listed below. First, the coefficient of $\frac{\tanh\tau}{\cosh^{\frac{4}{p}+4}\tau}$ gives the amplitude of the soliton as

$$A = \left[\frac{b^2}{ac}\frac{(p+1)(p+2)^2(p+4)}{\left(p^2+4p+8\right)^2}\right]^{\frac{1}{p}}. \tag{10}$$

Subsequently the coefficient of $\frac{\tanh\tau}{\cosh^{\frac{4}{p}+2}\tau}$ gives the inverse width of the soliton as

$$B = \frac{p}{2}\sqrt{\frac{b}{c\left(p^2+4p+8\right)}}. \tag{11}$$

Finally, the coefficient of $\frac{\tanh\tau}{\cosh^{\frac{4}{p}}\tau}$ yields the velocity of the soliton as

$$v = -\frac{4b^2}{c}\frac{p^2+4p+4}{\left(p^2+4p+8\right)^2}. \tag{12}$$

Thus, from (10) and (11), the amplitude and the inverse width of the soliton are related as

$$A^p = \frac{16cB^4}{ap^4}(p+1)(p+2)^2(p+4). \tag{13}$$

Thus, finally the one-soliton solution of the gKE is given by (4), where the amplitude, inverse width and the velocity of the soliton are respectively given by (10)–(12).

3. Integrals of motion

The gKE has at least two conserved quantities or integrals of motion [1]. They are the linear momentum (M) and energy (E). These are respectively given by

$$M = \int_{-\infty}^{\infty} q\,dx = \frac{A}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)} \tag{14}$$

and

$$E = \int_{-\infty}^{\infty} q^2\,dx = \frac{A^2}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{p}\right)}. \tag{15}$$

These conserved quantities are calculated by using the one-soliton solution given by (4). The center of the soliton $\bar{x}$ is given by the definition

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xq\mathrm{d}x}{\int_{-\infty}^{\infty} q\mathrm{d}x} = \frac{\int_{-\infty}^{\infty} xq\mathrm{d}x}{M}. \tag{16}$$

Thus, the velocity of the soliton is given by

$$v = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \frac{\int_{-\infty}^{\infty} xq_t\mathrm{d}x}{\int_{-\infty}^{\infty} q\mathrm{d}x} = \frac{\int_{-\infty}^{\infty} xq_t\mathrm{d}x}{M} \tag{17}$$

that leads to (12).

4. Conclusions

In this work, the travelling wave ansatz is used to integrate the gKE that serves as a generalized version of the KE and the mKE which were already studied, in a fairly detailed fashion, in the literature of solitons and integrability. This method is simple as compared to the involved IST that is also used to integrate nonlinear evolution equations. Unfortunately, the IST cannot be applied to integrate gKE as the Painlevé test of integrability will fail.

In future, this solution will be used to obtain the adiabatic parameter dynamics of the solitons in the presence of the perturbation terms. Also, the perturbed gKE will be integrated by the method of multiple scales and thus the quasi-stationary solitons will be obtained in the presence of these perturbation terms. Finally, the stochastic perturbation terms will be taken into account.

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